

Modelowanie preferencji a ryzyko'21

**Pareto Efficiency of Mixed Quantum Strategy
Equilibria in
Frąckiewicz-Pykacz parametrization
of EWL model**

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Games and probability distributions

We consider two player games

$$G = \left(N, \{S_X\}_{X \in N}, \{P_X\}_{X \in N} \right)$$

where:

$N = \{A, B\}$ is the set of players

$S_A = \{A_0, A_1\}, S_B = \{B_0, B_1\}$ are possible pure strategies

$P_X: S_A \times S_B \rightarrow \{v_{ij}^X \in \mathbb{R} \mid i, j = 0, 1\}, X = A, B$, are payoff functions, represented by the game bimatrix

$$\begin{pmatrix} (v_{00}^A, v_{00}^B) & (v_{01}^A, v_{01}^B) \\ (v_{10}^A, v_{10}^B) & (v_{11}^A, v_{11}^B) \end{pmatrix}$$

Let

$$\Delta(S_A \times S_B) = \left\{ \sum_{i,j=0,1} \sigma_{ij} A_i B_j \mid \sigma_{ij} \geq 0, \sum_{i,j=0,1} \sigma_{ij} = 1 \right\}$$

be the set of *probability distributions* over $S_A \times S_B$

Mixed strategies and Nash equilibria

If the set of probability distributions can be factorized

$$\begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{pmatrix} = \begin{pmatrix} \sigma_A \sigma_B & \sigma_A (1 - \sigma_B) \\ (1 - \sigma_A) \sigma_B & (1 - \sigma_A) (1 - \sigma_B) \end{pmatrix}$$

they define mixed strategies $\sigma_A, \sigma_B \in [0,1]$.

The *mixed classical game* is

$$G^{mix} = (N, \Delta S_A, \Delta S_B, \Delta P_A, \Delta P_B)$$

where $\Delta(S_X) = \{\sigma_X X_0 + (1 - \sigma_X) X_1 \mid 0 \leq \sigma_X \leq 1\} \equiv [0,1]$.

Mixed strategies form a subset of all probability distributions

$$\Delta S_A \times \Delta S_B \subset \Delta(S_A \times S_B)$$

The pair of strategies $(\sigma_A^*, \sigma_B^*) \in \Delta S_A \times \Delta S_B$ is a Nash equilibrium iff

$$\Delta P_A(\sigma_A^*, \sigma_B^*) \geq \Delta P_A(\sigma_A, \sigma_B^*) \quad \text{and} \quad \Delta P_B(\sigma_A^*, \sigma_B^*) \geq \Delta P_B(\sigma_A^*, \sigma_B),$$

for each $\sigma_X \in \Delta S_X, X = A, B$

Pareto optimality and correlated equilibria

A pair of strategies $(\sigma_A, \sigma_B) \in S$ is not Pareto optimal in S if there exists another pair $(\sigma_A', \sigma_B') \in S$ that is better for one of the players and not worse for the other. Otherwise $(\sigma_A, \sigma_B) \in S$ is called *Pareto optimal*.

Probability distribution $\{\sigma_{ij}\}_{i,j=0,1}$ over set of strategies $(A_i, B_j)_{i,j=0,1}$ of the game G is a *correlated equilibrium* iff

$$\sum_{j=0,1} \sigma_{ij} v_{ij}^A \geq \sum_{j=0,1} \sigma_{ij} v_{-ij}^A \quad \text{and} \quad \sum_{j=0,1} \sigma_{ji} v_{ji}^B \geq \sum_{j=0,1} \sigma_{ji} v_{j(-i)}^B$$

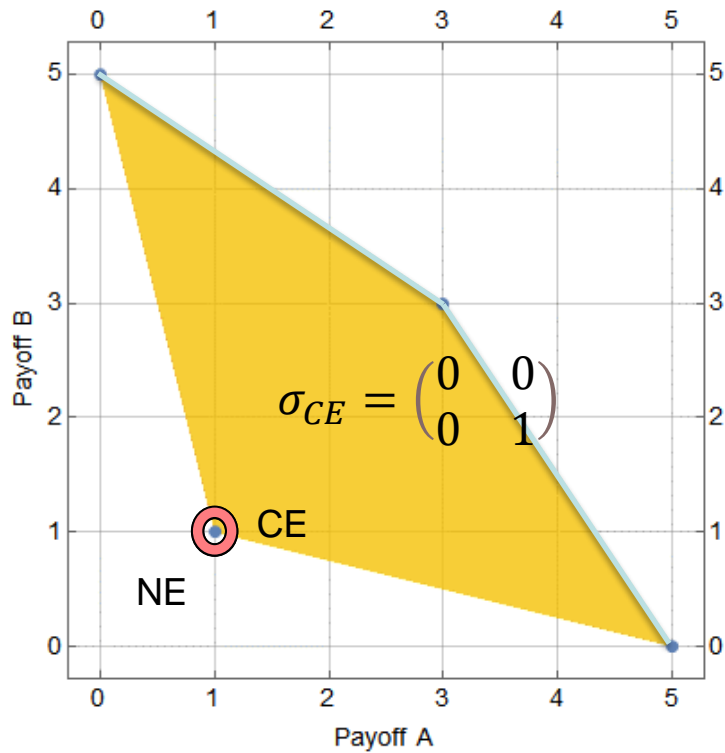
where $-i \neq i$ is the index if the remaining strategy.

Efficiency of selected classical games

		Bob	
		B_0	B_1
Alice	A_0	(3, 3)	(0, 5)
	A_1	(5, 0)	(1, 1)

$$\sigma_{00} = \sigma_{01} = \sigma_{10} = 0$$

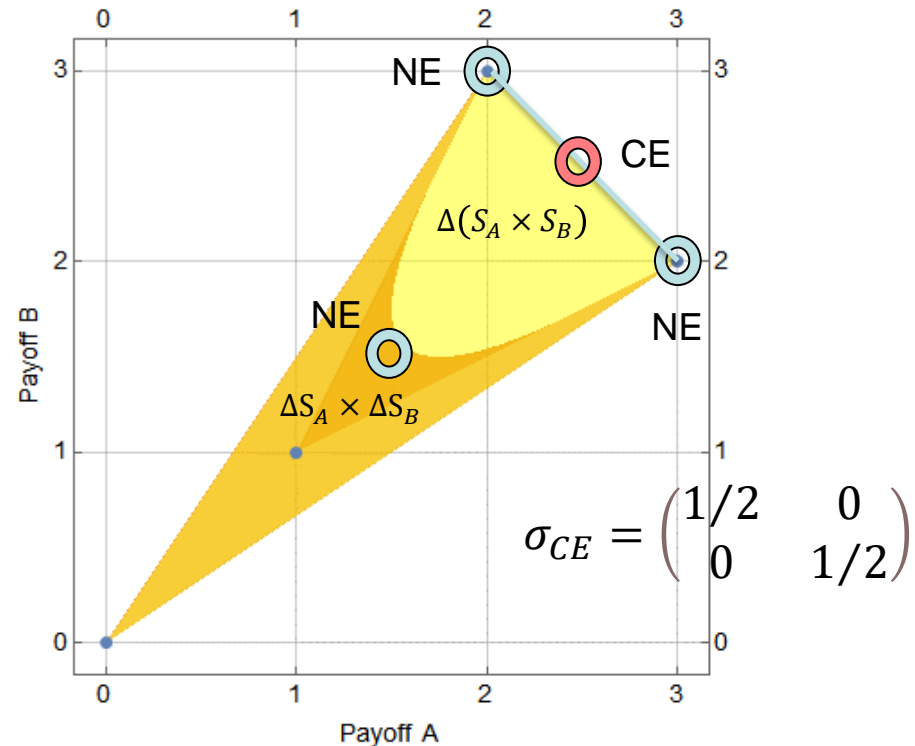
$$\sigma_{11} = 1$$



		Bob	
		B_0	B_1
Alice	A_0	(3, 2)	(1, 1)
	A_1	(0, 0)	(2, 3)

$$3\sigma_{00} \geq \sigma_{01}, \sigma_{00} \geq 3\sigma_{10}$$

$$3\sigma_{11} \geq \sigma_{01}, \sigma_{11} \geq 3\sigma_{10}$$



Efficiency of selected classical games

		Driver B	
		B_0	B_1
Driver A	A_0	(0, 0)	(0, 1)
	A_1	(1, 0)	(-10, -10)

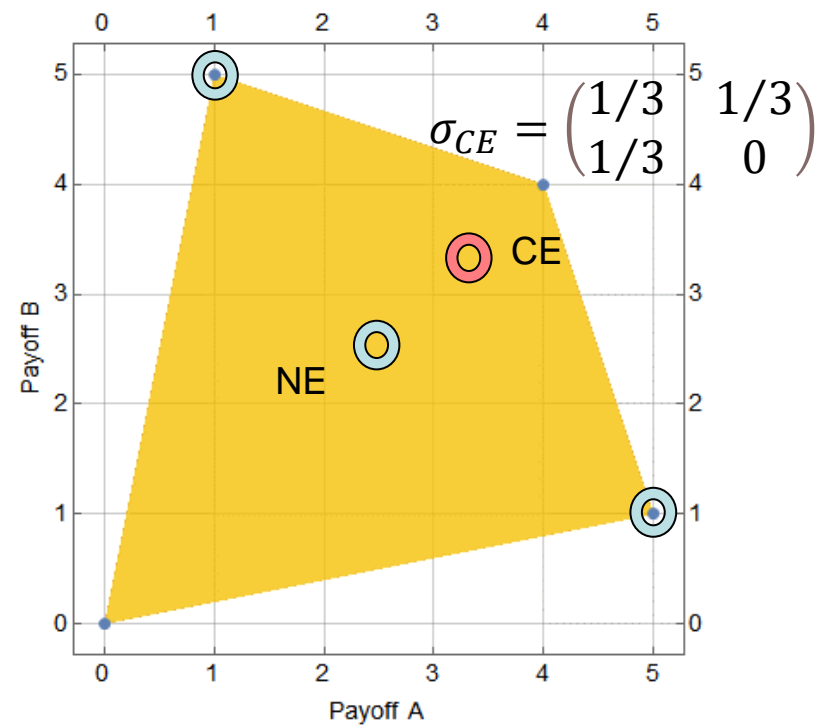
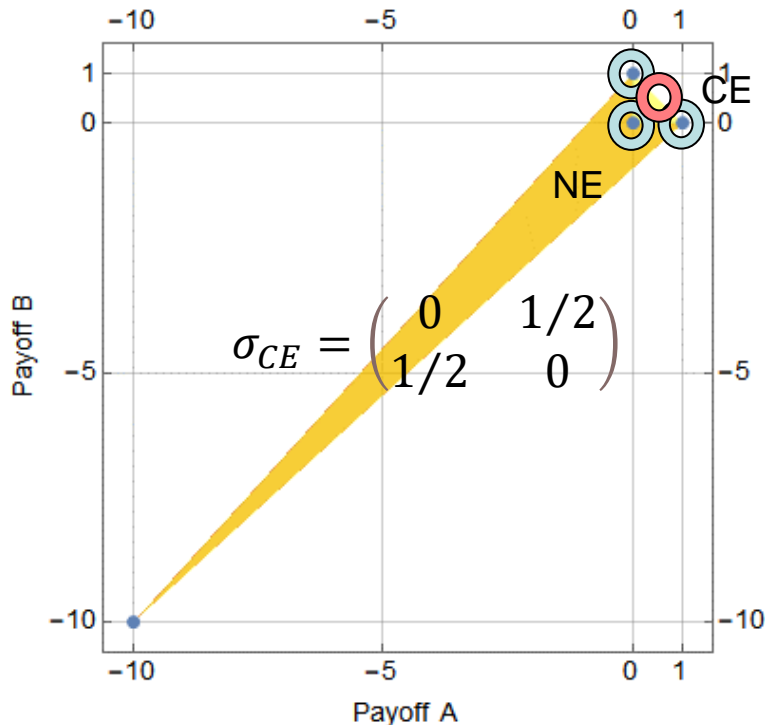
		Player B	
		B_0	B_1
Player A	A_0	(4, 4)	(1, 5)
	A_1	(5, 1)	(0, 0)

$$\sigma_{00} \leq 10\sigma_{01}, \sigma_{00} \leq 10\sigma_{10}$$

$$10\sigma_{11} \leq \sigma_{01}, 10\sigma_{11} \leq \sigma_{10}$$

$$\sigma_{00} \leq \sigma_{01}, \sigma_{00} \leq \sigma_{10}$$

$$\sigma_{11} \leq \sigma_{01}, \sigma_{11} \leq \sigma_{10}$$



Quantum games

The quantum game in Eisert-Wilkens-Lewenstein quantization scheme is: (Eisert et al, PRL 83, 3077 (1999))

$$\Gamma_{EWL} = (N, \{U_X\}_{X \in N}, \{\Pi_X\}_{X \in N})$$

where:

$N = \{A, B\}$ is the set of players

$\hat{U}_A = \hat{U}(\theta_A, \alpha_A, \beta_A)$, $\hat{U}_B = \hat{U}(\theta_B, \alpha_B, \beta_B)$, are unitary transformations (quantum strategies)

$\Pi_X: SU(2) \times SU(2) \rightarrow \mathbb{R}$ are payoff functions:

$$\Pi_X(\hat{U}_A, \hat{U}_B) = \sum_{i,j=0,1} |p_{ij}|^2 v_{ij}^X, \quad X = A, B, \text{ where}$$

$$|p_{00}|^2 = \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \cos(\alpha_A + \alpha_B) + \sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \sin(\beta_A + \beta_B),$$

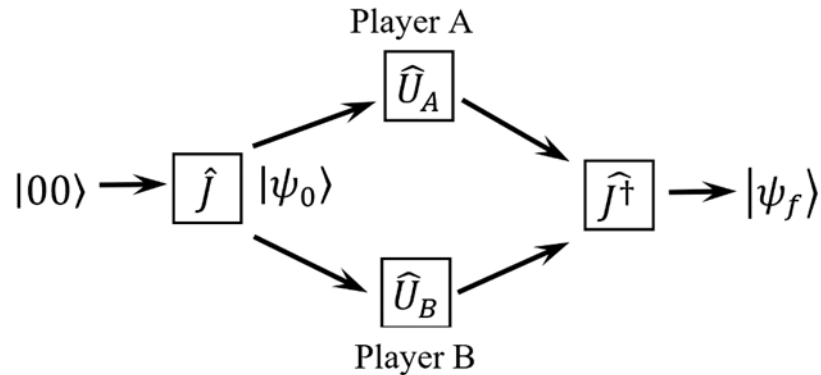
$$|p_{01}|^2 = \cos \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \cos(\alpha_A - \beta_B) + \sin \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \sin(\alpha_B - \beta_A),$$

$$|p_{10}|^2 = \cos \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \sin(\alpha_A - \beta_B) + \sin \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \cos(\alpha_B - \beta_A),$$

$$|p_{11}|^2 = \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \sin(\alpha_A + \alpha_B) - \sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \cos(\beta_A + \beta_B).$$

EWL approach

The quantum EWL approach to the game is



where: $|00\rangle$ is the initial state

$\hat{J} = \frac{1}{\sqrt{2}}(\hat{I} + i\sigma_x \otimes \sigma_x)$, \hat{J}^\dagger are the entangling, disentangling operators,

$$\hat{U}_X(\theta_X, \alpha_X, \beta_X) = \begin{pmatrix} e^{i\alpha_X} \cos \frac{\theta_X}{2} & ie^{i\beta_X} \sin \frac{\theta_X}{2} \\ ie^{-i\beta_X} \sin \frac{\theta_X}{2} & e^{-i\alpha_X} \cos \frac{\theta_X}{2} \end{pmatrix}, X = A, B,$$

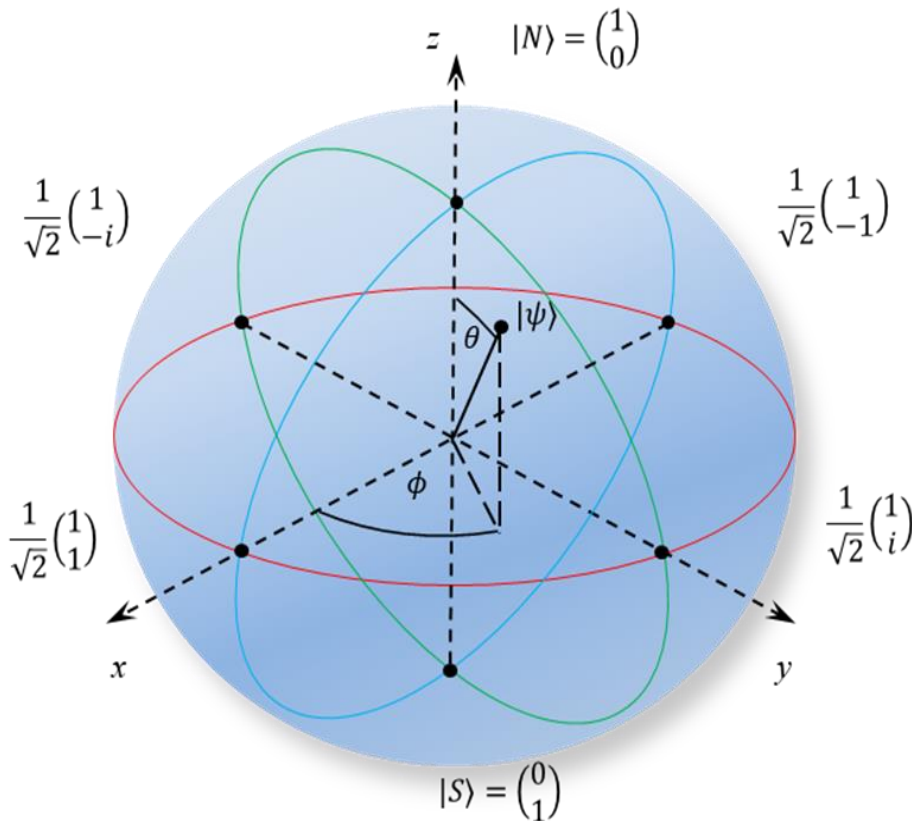
$|\psi_f\rangle = \sum_{i,j=0,1} p_{ij} |ij\rangle$, is the final state defining the game payoffs

Quantum strategies

Strategies $\hat{U}_A = \hat{U}(\theta_A, \alpha_A, \beta_A)$ and $\hat{U}_B = \hat{U}(\theta_B, \alpha_B, \beta_B)$,

$$\hat{U}_X(\theta_X, \alpha_X, \beta_X) = \begin{pmatrix} e^{i\alpha_X} \cos \frac{\theta_X}{2} & ie^{i\beta_X} \sin \frac{\theta_X}{2} \\ ie^{-i\beta_X} \sin \frac{\theta_X}{2} & e^{-i\alpha_X} \cos \frac{\theta_X}{2} \end{pmatrix}, \text{ are generated by Pauli strategies:}$$

$$\begin{aligned} \hat{P}_0 &= \hat{U}(0, 0, \beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \hat{P}_x &= \hat{U}(\pi, \alpha, \pi) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \\ \hat{P}_y &= \hat{U}(\pi, \alpha, \pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \hat{P}_z &= \hat{U}(0, \pi/2, \beta) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned}$$



where

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

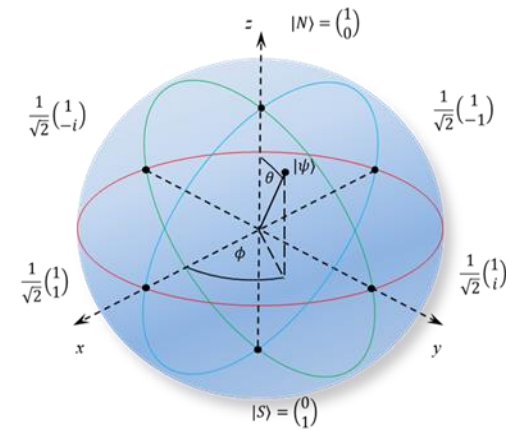
are Pauli matrices

Classical limit of the quantum game

Let us assume $\alpha = \beta = 0$, in this case

$$\hat{U}(\theta, 0, 0) = \cos \frac{\theta}{2} \hat{I} + i \sin \frac{\theta}{2} \sigma_x$$

is equivalent to the classical mixed strategy



		B	
		$\theta_B = 0$	$\theta_B = \pi$
A	$\cos^2 \frac{\theta_A}{2}$	(a_{00}, b_{00})	(a_{01}, b_{01})
	$\sin^2 \frac{\theta_A}{2}$	(a_{10}, b_{10})	(a_{11}, b_{11})

and the payoff is

$$\begin{aligned} \$_{A(B)} = & a(b)_{00} \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} + a(b)_{01} \cos^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} \\ & + a(b)_{10} \sin^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} + a(b)_{11} \sin^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} \end{aligned}$$

EWL with Frąckiewicz-Pykacz parameterization

Let us restrict the set of quantum strategies to

$$\widehat{U}_X(\theta_X, \phi_X) = \begin{pmatrix} e^{-i\phi_X} \cos \frac{\theta_X}{2} & -e^{-i\phi_X} \sin \frac{\theta_X}{2} \\ e^{i\phi_X} \sin \frac{\theta_X}{2} & e^{i\phi_X} \cos \frac{\theta_X}{2} \end{pmatrix}$$

$$\begin{aligned} \widehat{P}_0 &= \widehat{U}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \widehat{P}_x &= \widehat{U}\left(\pi, \frac{3\pi}{2}\right) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \\ \widehat{P}_y &= \widehat{U}(\pi, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \widehat{P}_z &= \widehat{U}\left(0, \frac{3\pi}{2}\right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned}$$

- In this parameterization, Nash equilibria in pure strategies are possible
- F-P parametrization is invariant with respect to strongly isomorphic transformation of input games

Quantum game in Pauli strategies

The payoff matrix of Pauli strategies in the EWL scheme

		Player B			
		\widehat{P}_0	\widehat{P}_x	\widehat{P}_y	\widehat{P}_z
Player A	\widehat{P}_0	(a_{00}, b_{00})	(a_{01}, b_{01})	(a_{10}, b_{10})	(a_{11}, b_{11})
	\widehat{P}_x	(a_{10}, b_{10})	(a_{11}, b_{11})	(a_{00}, b_{00})	(a_{01}, b_{01})
	\widehat{P}_y	(a_{01}, b_{01})	(a_{00}, b_{00})	(a_{11}, b_{11})	(a_{10}, b_{10})
	\widehat{P}_z	(a_{11}, b_{11})	(a_{10}, b_{10})	(a_{01}, b_{01})	(a_{00}, b_{00})

one can construct mixed Pauli strategies defined by quadruples of coefficients:

$$\Delta U_X \equiv \Delta(U_X) = \left\{ \sum_{\alpha=0,x,y,z} \sigma_\alpha^X \widehat{P}_\alpha \mid 0 \leq \sigma_\alpha^X; \sum_{\alpha=0,x,y,z} \sigma_\alpha^X = 1 \right\}, \quad X = A, B,$$

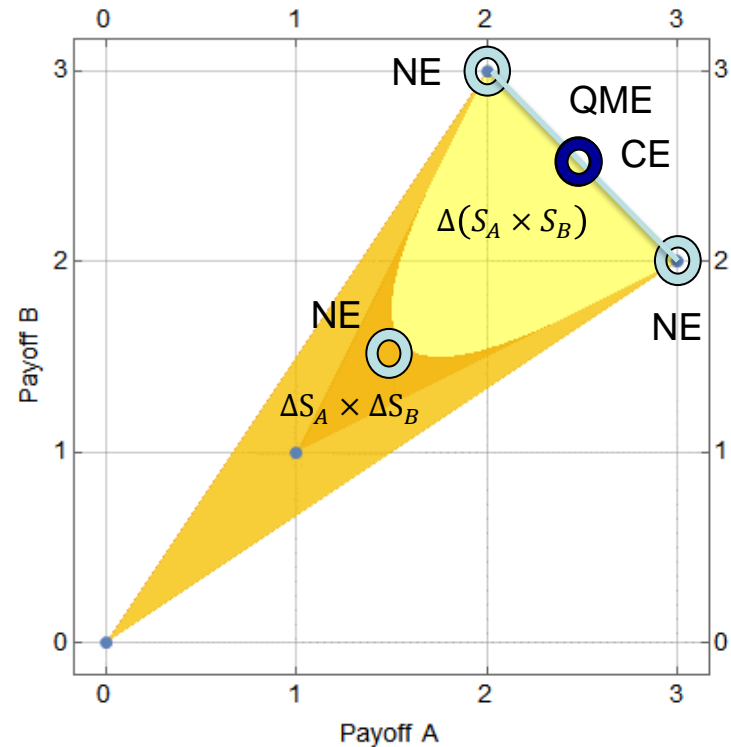
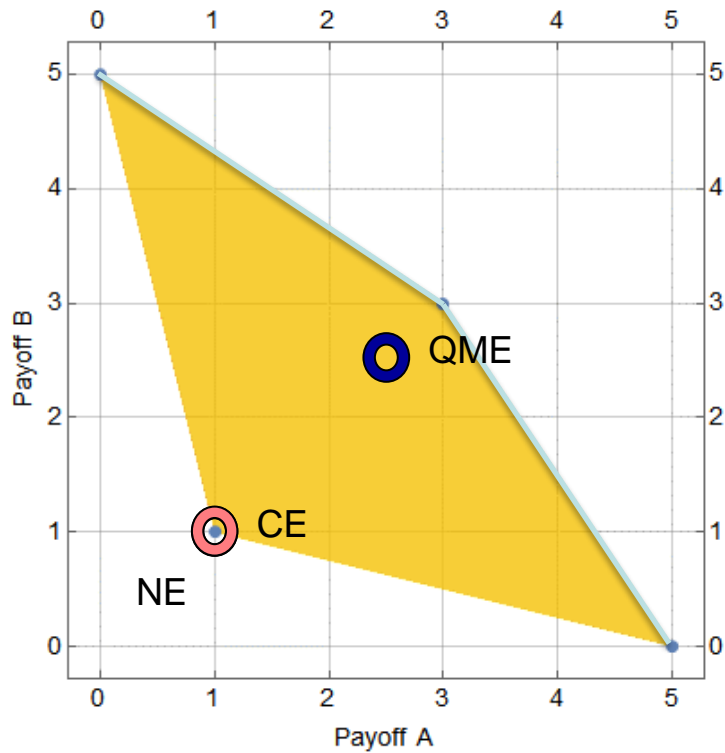
Quantum Mixed Equilibria

prisoner's dilemma		Bob	
		B_0	B_1
Alice	A_0	(3, 3)	(0, 5)
	A_1	(5, 0)	(1, 1)

battle of the sexes		Bob	
		B_0	B_1
Alice	A_0	(3, 2)	(1, 1)
	A_1	(0, 0)	(2, 3)

$$\sigma^A = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right), \sigma^B = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\sigma^A = \sigma^B = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

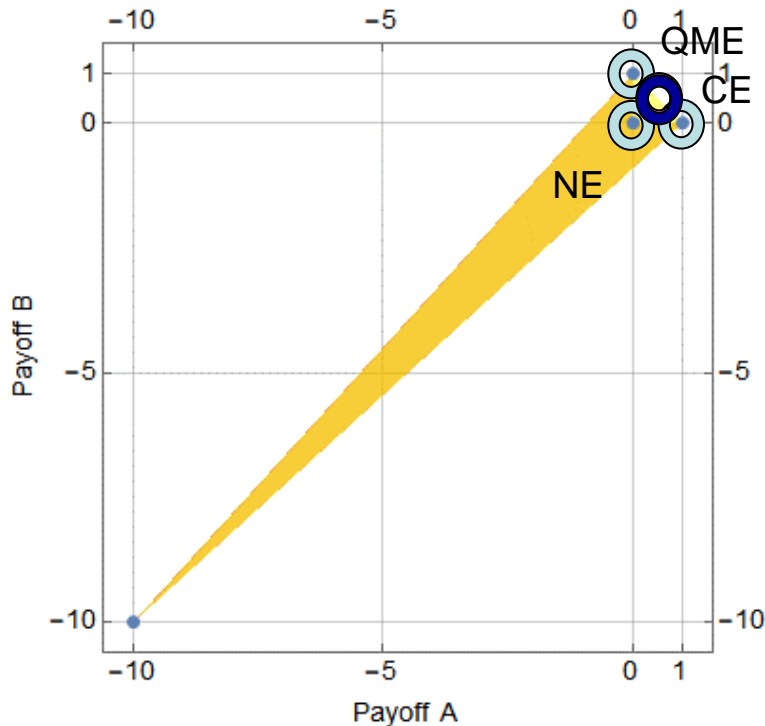


Quantum Mixed Equilibria

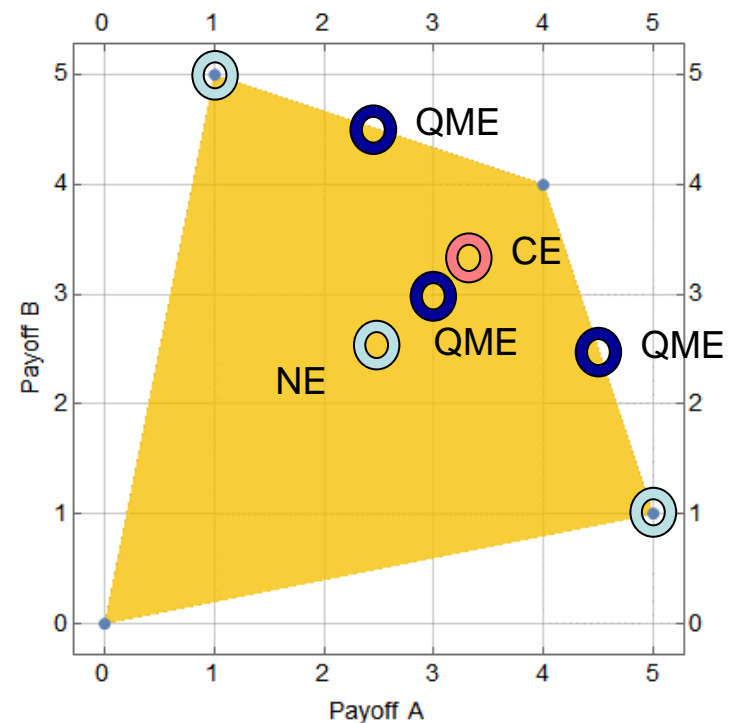
		Driver B	
		chicken	Driver B
Driver A		B_0	B_1
	A_0	(0, 0)	(0, 1)
	A_1	(1, 0)	(-10, -10)

		Player B	
		chicken 2	Player B
Player A		B_0	B_1
	A_0	(4, 4)	(1, 5)
	A_1	(5, 1)	(0, 0)

$$\sigma^A = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right), \sigma^B = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$$

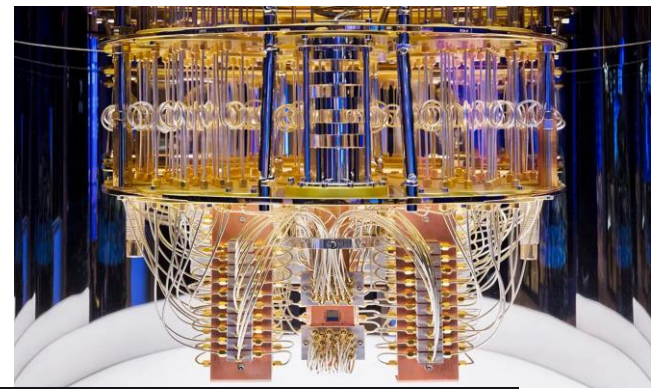


$$\sigma^A = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \sigma^B = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$$



Quantum Computer

<https://quantum-computing.ibm.com/>



The screenshot displays the IBM Quantum Experience web interface. At the top, the title "IBM Quantum Experience" is visible. Below it is a navigation menu with options: File, Edit, Inspect, View, Share, and Help. The current project is titled "Circuits / Untitled circuit", and there is a "Simulator seed" link on the right. A toolbar contains various quantum gates: Hadamard (H), CNOT, Toffoli, SWAP, Identity (I), T, S, Z, T†, S†, P, RZ, a black circle, |0>, and a Z rotation gate. Below the toolbar is another row of gates: if, a vertical ellipsis, square root of X (√X), square root of X† (√X†), Y, RX, RY, U, RXX, RZZ, and a "+ Add" button. The main workspace shows a quantum circuit with four qubits: q0, q1, q2, and c3. Qubits q0, q1, and q2 are represented by horizontal lines, while c3 is a classical control line. The circuit includes CNOT gates between q0 and q1, and q1 and q2. There are also CNOT gates from q0 to q2 and q1 to q2. A Z rotation gate is applied to q0, and a measurement gate is applied to q0. The measurement result is stored in the classical register c3, which is labeled with "0". Another measurement gate is applied to q1, and its result is stored in the classical register c3, which is labeled with "1". On the right side of the circuit, there are three circular indicators representing the state of the qubits: the top one is grey, the middle one is blue, and the bottom one is blue.

Conclusions

1. In some games the set of mixed strategies is a proper subset of the set of probability distributions
2. Correlated equilibria significantly improve paretoefficiency of Nash equilibria
3. Quantum games give players new strategies not available in classic games and strongly depend on the parameterization used
4. Nash equilibria of quantum in mixed strategies are close to paretoefficiency of correlated equilibria
5. FP parameterization provides a strong isomorphism of the quantum game and gives the same Nash equilibria in mixed strategies as full $SU(2)$ parameterization of EWL